Group Theory 12

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Conjugates :-

- If G is a group, then the relation "y conjugate of x in G" means, $y = gxg^{-1}$ for some $g \in G$, is an equivalency class.

- If G is a group, then the equivalency class $a \in G$ under the relation "y conjugate of x in G" is called conjugacy class which is denoted by a^G

Center :-

Def :- The center of a group G is denoted by $Z(G)$ is the set of all $a \in G$ that commute with every element in G

- $Z(G)$ is a normal abelian subgroup of G

If $a \in Z(G)$, $\quad ag = ga \quad \forall g \in G$

$\qquad a = gag^{-1} \quad \forall a \in Z(G)$

Thus $Z(G)$ is normal.

Def :- If $a \in G$, then the centralizer of a in G denoted by $C_G(a)$ is the set of all $x \in G$ which commute with a

- C_G is a subgroup of G.

Theorem :- If $a \in G$, the number of conjugates of a is equal to the index of its centralizer, i.e,

$$|a^G| = [G : C_G(a)]$$

We also have $|a^G| \mid |G|$ when G is finite

Proof :- $f : a^G \to G/C_G(a)$

$\qquad gag^{-1} \to gC_G(a)$

Proof:- $f : a^G \to G/C_G(a)$

$gag^{-1} \to gC_G(a)$

$gag^{-1} = hah^{-1} \Rightarrow h^{-1}gag^{-1}h = a \Rightarrow h^{-1}ga = ah^{-1}g$

$h^{-1}g$ commutes with a

$\Rightarrow h^{-1}g \in C_G(a) \Rightarrow hC_G(a) = gC_G(a)$

So we get f is injective

$gC_G(a) \in G/C_G(a)$

Then we get $f(gag^{-1}) = gC_G(a)$

So f is surjective

So f is bijective. $\Rightarrow |a^G| = [G : C_G(a)]$

Definition:- If $H \leq G$ and $g \in G$ then the conjugate gHg^{-1} which is often denoted by H^g is $\{ghg^{-1} : h \in H\}$

Definition:- If $H \leq G$ then the normalizer of H in G denoted by $N_G(H)$ is $\{a \in G : aHa^{-1} = H\}$

- $N_G(H)$ is a subgroup of G.
- $H \trianglelefteq N_G(H)$

Theorem:- If $H \leq G$ then the number of conjugates of H in G is equal to the index of its normalizer, i.e., $c = [G : N_G(H)]$ and $c \mid |G|$ when G is finite.

in G is equal to the ...

$c = [G : N_G(H)]$ and $c \mid |G|$ when G is finite.

Also, $aHa^{-1} = bHb^{-1}$ iff $b^{-1}a \in N_G(H)$.

Q> A group is centerless if $Z(G) = \{1\}$. Prove that S_n is centerless if $n \geq 3$.

Ans:- $(a\ b)(a\ c) = (a\ c\ b)$

$(a\ c)(a\ b) = (a\ b\ c)$

$(a\ b\ c \ldots)(b\ d) = (a\ b \quad)$

$(b\ d)(a\ b\ c\ . \quad) = (a\ d \ldots)$

Q> If $\alpha \in S_n$ is a n-cycle then it's centralizer is $\langle \alpha \rangle$. Prove it.

Ans:- $y \in C_G(\alpha) \setminus \langle \alpha \rangle$

$y\alpha = \alpha y$

$y = \alpha y \alpha^{-1}$

$\alpha = y \alpha y^{-1}$

not true for different permutations, i.e., $y \neq \alpha^k$ form

$\alpha^k \alpha = \alpha \alpha^k$

$\alpha^k \in C_G(\alpha)$

$\langle \alpha \rangle \in C_G(\alpha)$

So we must have $y = \alpha^k$ form. So $\Rightarrow \Leftarrow$

So we get $\langle \alpha \rangle = C_G(\alpha)$

Definition:- Two permutations $\alpha, \beta \in S_n$ have the same cycle structure if their complete factorization into disjoint cycles have the same number of r-cycles for ...

disjoint cycles have ..
each r.

Lemma:- If $\alpha, \beta \in S_n$ then $\alpha\beta\alpha^{-1}$ is the permutation with the same cycle structure as β which is obtained by applying α to β

$\beta = (1\ 3)(2\ 4\ 7)$, $\alpha = (2\ 5\ 6)(1\ 4\ 3)$

$\alpha\beta\alpha^{-1} = (4\ 1)(5\ 3\ 7)$

Theorem:- Permutation $\alpha, \beta \in S_n$ are conjugate iff they have the same cycle structure

Proof:- α and β are conjugate

$\alpha = g\beta g^{-1}$ By previous lemma done

Converse, α, β have same cycle structure

$\alpha = \alpha_1 \alpha_2 \cdot \quad \alpha_n$ $\quad \alpha_i$'s are disjoint

$\beta = \beta_1 \beta_2 \ldots \beta_n$ $\quad \beta_i$'s are disjoint

Take one-one correspondence to α_i to β_i and take complete factorization

$\alpha(x_1) = y_1$

$\beta(x_2) = y_2$

Define, $\gamma \in S_n$ such that,

$\gamma(x_1) = x_2$ and $\gamma(y_1) = y_2$

$(\gamma\alpha\gamma^{-1}(x_2)) = (\gamma\alpha(x_1)) = \gamma(y_1) = y_2 = \beta(x_2)$

$$\left(\gamma\alpha\gamma^{-1}(x_2)\right) = \left(\gamma\alpha(x_1)\right) = \gamma(y_1) = y_2 = \beta(x_2)$$

$$\beta = \gamma\alpha\gamma^{-1}$$

$\Rightarrow \alpha, \beta$ are conjugate

- A subgroup H of S_n is a normal subgroup iff whenever $\alpha \in H$, then every β having the same cycle structure as α also belongs to H.